Module 14 - Related Rates

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Introduction

In this module you will solve problems in which two or more related variables change over time. Since the variables are related, their rates of change are also related. Therefore, if you are given one of the rates of change you should be able to find the other rate of change.

Lesson Index:

14.1 - The Ladder Problem

14.2 - The Ladder Problem With Gravity

14.3 - Two Ships

14. 4 - A Moving Particle

After completing this module, you should be able to do the following:

- Use the TI-89 to define functions with variables that change over time
- Differentiate these functions with respect to time
- Solve for related rates of change
- Model related rate problems parametrically

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Lesson 14.1: The Ladder Problem

This lesson explores related rates by investigating the positions of the foot and the top of a ladder as it slides down a wall.

A 15 foot ladder is held against a wall and then released. The foot of the ladder begins to slide along the ground away from the wall at a constant rate of 3 ft/sec. In the diagram below, the distance from the wall to the foot of the ladder is represented by x, and y represents the distance from the ground to the top of the ladder.

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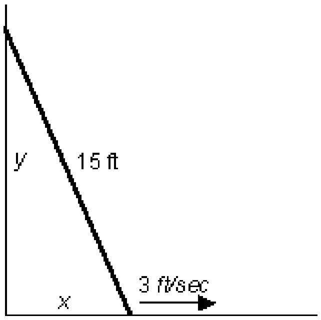


- Execute NewProb
- Set the Graph mode to PARAMETRIC

Modeling the Position of the Ladder's Foot

Parametric equations can be used to model the position of the foot of the ladder.

Because the foot of the ladder is moving at a constant rate of 3 ft/sec, it follows that at time t the ladder is $3t$ from the wall. That is,

$x = 3t$

Because the foot of the ladder does not rise above the ground and its position along the x-axis is given by $3t$, the position of the foot of the ladder can be modeled by the parametric equations given below.

- Set xt1 = 3t
- Set yt1 = 0

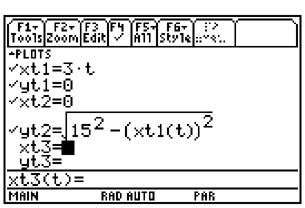
Modeling the Position of the Ladder's Top

An expression for the position of the ladder's top, y, can by found by using the Pythagorean theorem.

$y = \sqrt{15^2 - x^2}$

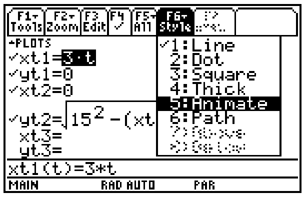
The position of the top of the ladder can be modeled by another set of parametric equations.

- Set xt2 = 0



Animating the Graphs

- Set the Graph Style of the equations to "Animate" by selecting an equation, opening the Style menu by pressing [2nd] [F6] and choosing "5:Animate"



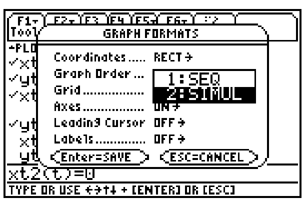
- Repeat the process for each pair of equations

Tech Tip!

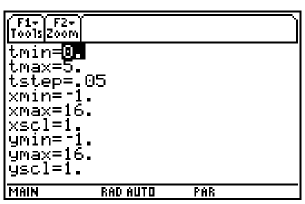
Setting Parametric Graph Style
The cursor can be on either *x* or *y* when changing the graphing style of a pair of parametric equations.

Simulating the Ladder's Motion

Graph the equations simultaneously.

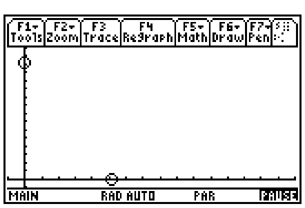


- Enter the following window values:



- Graph the equations

You should see animated circles that simulate the movement of the top and bottom of the ladder.



You can regraph the equations by pressing [F4].

14.1.1 Describe the rates at which the top and bottom of the ladder move. Click here for the answer.

Answer 1

14.1.1 The bottom of the ladder moves at a constant rate of 3 ft/sec while the top of the ladder falls at an increasing rate.

Tech Tip!

Changing the Animation Speed
You can change the apparent speed of the simulation with the value of *tstep* in the Window editor. Smaller values of *tstep* slow the simulation, larger values speed it up.

Finding the Velocity of the Ladder's Top

How fast is the top moving down the wall when the bottom is 5 ft from the wall?

The instantaneous velocity of the top of the ladder is the value needed, and it is given by the derivative of the function that represents the position of the top. To find the velocity of the top when the bottom is 5 ft from the wall, evaluate the derivative of the position of the top given by yt2 at the value of t that corresponds to $x = 5$.

Because $x = 3t$, $x = 5$ corresponds to $t = \frac{5}{3}$. Find the value of $\frac{dy}{dt}$ when $t = \frac{5}{3}$.

- Evaluate *d*(yt2(t),t) | t=5/3

14.1.2 How fast is the top of the ladder moving when the bottom is 5 ft from the wall? Click here for the answer.

Answer 2

14.1.2 $\frac{dy}{dt} = \frac{-3\sqrt{2}}{4} \approx -1.061$ ft/sec . The negative sign indicates that y is decreasing.

Finding the Velocity Function for the Ladder's Top

The speed of the top of the ladder is a function of time, which is the derivative of its position function.

- Evaluate *d*(yt2(t),t)

14.1.3 What is the velocity function for the ladder's top? Click here for the answer.

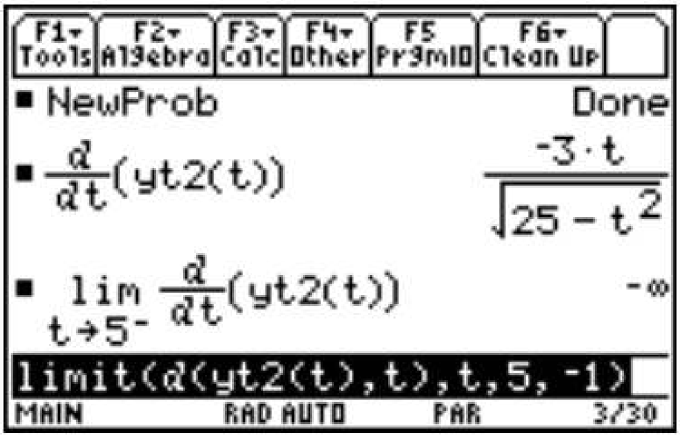
Answer 3

14.1.3 The velocity of the ladder's top is given by $\frac{dy}{dt} = \frac{-3t}{\sqrt{25-t^2}}$

14.1.4 How fast is the top of the ladder moving, in theory, when it hits the ground? Click here for the answer.

Answer 4

14.1.4 Solving yt2(t)=0 indicates that the top of the ladder hits the ground after 5 seconds, but $\frac{dy}{dt} = \frac{-3t}{\sqrt{25-t^2}}$ is undefined at that time. As t approaches 5 seconds, $\frac{dy}{dt}$ grows without bound. So in theory the top is going infinitely fast when it hits the ground as suggested by the following figure.



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Lesson 14.2: The Ladder Problem With Gravity

The ladder problem investigated in Lesson 14.1 is extended in this lesson to include the force of gravity on the ladder as it slides down the wall.

Suppose the fifteen-foot ladder from Lesson 14.1 is held against a wall and the top of the ladder is being pulled downward by the force of gravity. The equation below models the position of the top of the ladder where the term $-16t^2$ is the motion of the top of the ladder due to gravity and 15 represents the position of the top at time $t = 0$ seconds.

$$y = -16t^2 + 15$$

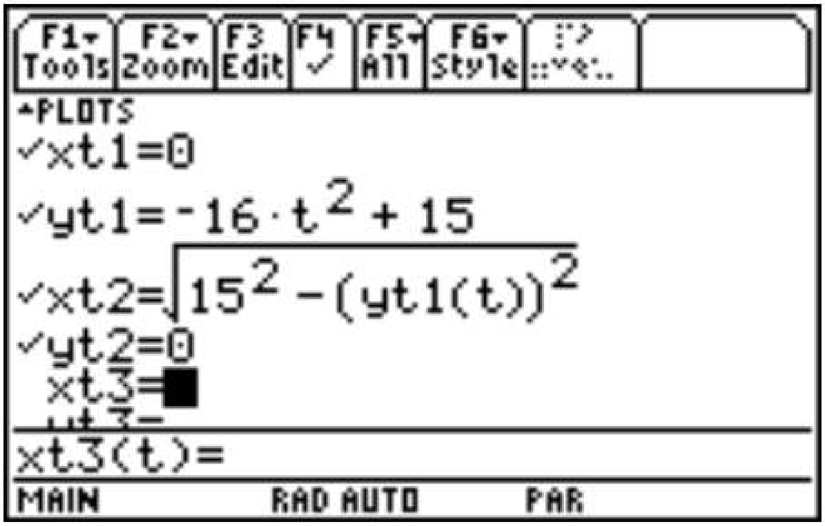
The equation for the position of the bottom of the ladder can be found by using the Pythagorean theorem.

$$x = \sqrt{15^2 - y^2}$$

14.2.1 What are the parametric equations that model the position of the top and bottom of the ladder at time t? Click here for the answer.

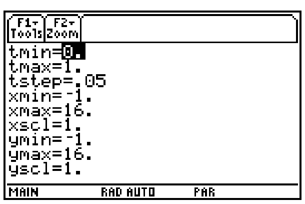
Answer 1

14.2.1 The parametric equations that model the ladder's position at time t are shown in the screen below.



Displaying the Model

- Enter the parametric equations found in Question 14.2.1 into the Y= Editor
- Choose "Animate" for the Graph Style for both pairs of equations
- Make sure the Graph Order is still "Simultaneous"
- Use the Viewing Window values shown below



- Graph the equations

14.2.2 Describe the apparent motion of the top and bottom of the ladder. Click here for the answer.

Answer 2

14.2.2 The top of the ladder is falling at an increasing rate while the bottom is moving at a decreasing rate.

14.2.3 How fast is the bottom of the ladder moving at $t = 0.2$, $t = 0.5$ and $t = 0.8$ seconds? Click here for the answer.

Answer 3

14.2.3 Using the derivative of the position of the bottom of the ladder, xt2(t), and evaluating it at the given times, the bottom is moving at approximately 21.2015 ft/sec, 17.2582 ft/sec, and 8.5665 ft/sec, respectively.

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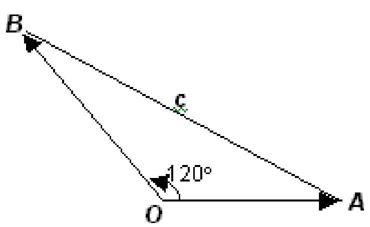
Lesson 14.3: Two Ships

*A related-rate problem that models **two ships** as they move away from each other is discussed in this lesson.*

Two ships start at a point O and move away from that point along routes that make a 120° angle. **Ship A** moves at 14 knotsand **Ship B** moves at 21 knots.

In the diagram below ***A*** represents the position of **Ship A**, ***B*** represents the position of **Ship B**, and ***c*** represents the distance between the ships.

(A knot is a unit used to measure the speed of a ship. One knot represents one nautical mile (6,076.1 feet) an hour).



Modeling the Positions of the Ships

Suppose that Ship A is moving along the positive x-axis at 14 knots. Enter the parametric equations for Ship A's position at time *t* hours.

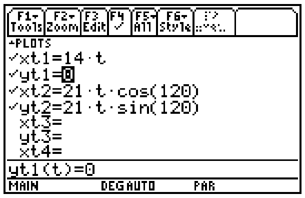
- Set xt1 = 14t
- Set yt1 = 0

Parametric equations can be used to model the position of an object at time *t* that is moving at speed *v* along a line that forms an angle θ with the positive x-axis. Such equations have the form

$x = vt\cos\theta$
$y = vt\sin\theta$

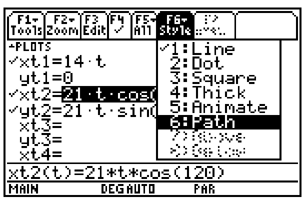
Enter the equations for Ship B's position, which is moving at 21 knots along a line that forms a 120° angle with the path of Ship A.

- Set Angle mode to DEGREE
- Set xt2 = 21t*cos(120)
- Set yt2 = 21t*sin(120)



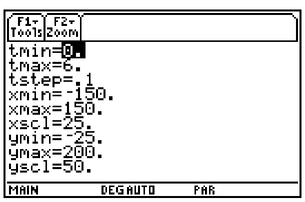
Animating the Motion of the Ships

- Change the Graph Style of both sets of equations to "Path"

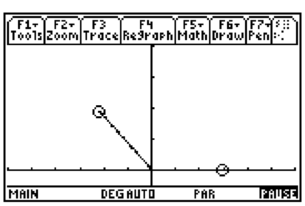


Make sure the Graph Order in the Graph Formats dialog box is still Simultaneous.

- Enter the Viewing Window values shown below:



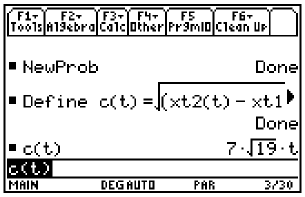
- Graph the equations



Finding the Distance between the Ships

The distance $d(t)$ between the ships can be found by using the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

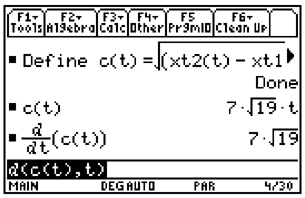
Use the restriction $t \geq 0$ because the problem starts at $t = 0$.



Thus, $c(t) = 7\sqrt{19}\,t$.

Finding the Speed at which the Ships are Moving Apart

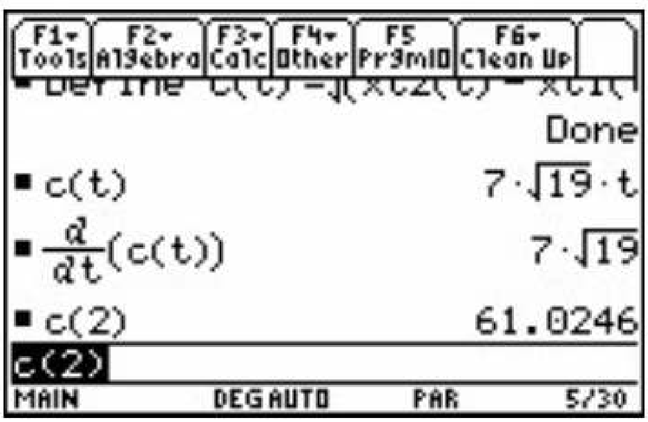
The speed at which the ships are moving apart can be found by finding the derivative of *c*(*t*) with respect to *t*, or just observing that it is $7\sqrt{19}$ knots from the formula for *c*(*t*).



14.3.1 Approximately how far apart are the ships after two hours? Click here for the answer.

Answer 1

14.3.1 About 61.0 nautical miles.



Modifying the Problem

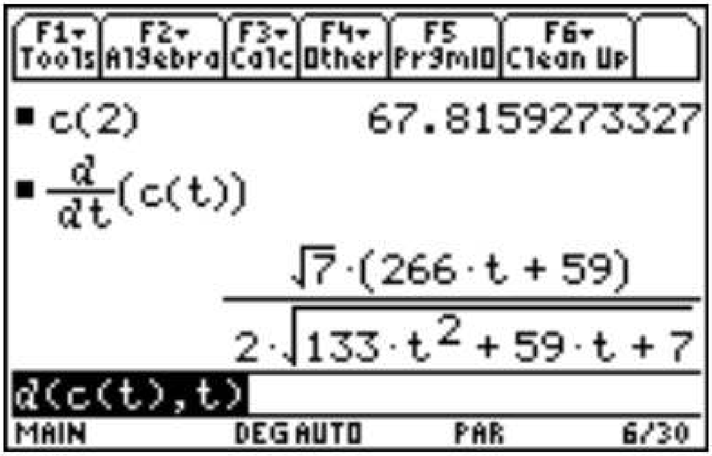
Assume that both ships travel in the same direction and at the same speed as before, but Ship A begins its journey 5 nautical miles from point O and Ship B begins 3 nautical miles from point O.

- Set xt1 with 14t+5
  xt2 with (21t+3)*cos(120)
  yt2 with (21t+3)*sin(120)

14.3.2 Find the derivative of *c*(*t*). Click here for the answer.

Answer 2

14.3.2



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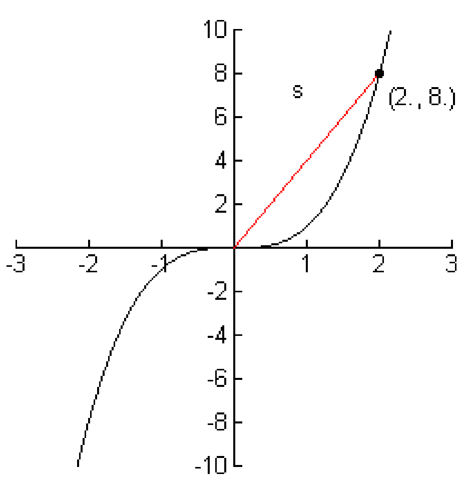
Lesson 14.4: A Moving Particle

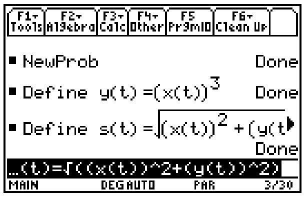
A final related-rate problem, which models a particle moving along a curve, is explored in this lesson.

A particle is moving along the curve $y = x^3$. At a certain instant, the particle is at the point (2, 8) and dx/dt = 5 ft/sec.
How fast is the distance *s* from the particle to the origin changing at that instant?



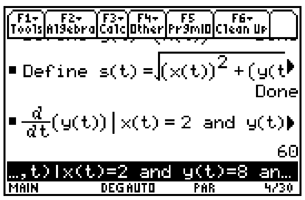
Find the Distance Function *s*(*t*)

- Execute the NewProb command
- Define y(t)=(x(t))^3



Find *dy/dt* at the Point (2, 8)

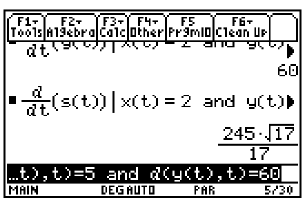
We know that *dx/dt* = 5 at this point. Thus, we need to compute *dy/dt* with the restrictions that x(t)=2, y(t)=8, and *d*(x(t),t)=5.



So, *dy/dt* = 60 ft/sec at the point (2, 8).

Answering the Question

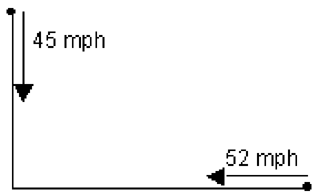
To find out how fast the distance *s* from the particle to the origin is changing at the specified instant, we need to compute *ds/dt* with the restrictions that x(t)=2, y(t)=8, dx/dt=5, and dy/dt=60.



Therefore, the particle is moving away from the origin at $\frac{245\sqrt{17}}{17} \approx 59.4212$ ft/sec.

Self Test

Two cars are approaching an intersection at the origin in the figure below. One car is 15 miles north of the intersection and traveling at a rate of 45 mph. The other car is 24 miles east of the intersection and traveling at a rate of 52 mph.



Q1. What are the parametric equations that represent the position of the car approaching from the north?
Q2. What are the parametric equations that represent the position of the car approaching from the east?
Q3. Select "Animate" graph Style and "Simultaneous" graph Order for the equations in Questions 1 and 2, and simulate the movement of the cars using the window below, and answer the question that follows.

tmin = 0	**xmin** = -5	**ymin** = -5
tmax = 0.4	**xmax** = 40	**ymax** = 20
tstep = 0.01	**xscl** = 5	**yscl** = 5

According to the simulation, which car reaches the intersection first?

^^-- end of **Q3** --^^

Q4. Define a function that represents the distance between the two cars.
Q5. How fast is the distance between the cars changing at $t = 0.25$ hours?

Click here to check your answers.

Answer 1

$x(t) = 0$

$y(t) = 15 – 45t$

Answer 2

$x(t) = 24 – 52t$

$y(t) = 0$

Answer 3

The car coming from the North will arrive at the intersection first.

Answer 4

Define c(t) = √((xt2(t))^2 + (yt1(t))^2)

Answer 5

$\frac{d}{dt}c(t)$ with t = 0.25 is approximately -63.739, so the distance between the cars is decreasing at a rate of 63.739 mph.

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